Geometric Composition in Quilted Floer Theory

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We prove that Floer cohomology of cyclic Lagrangian correspondences is invariant under transverse and embedded composition under a general set of assumptions. We also give an application of this result in the negatively monotone setting to construct an isomorphism in Floer theory of broken fibrations.

1 Introduction

1.1 Lagrangian correspondences and geometric composition

Given two symplectic manifolds (M_1, ω_1), (M_2, ω_2) a Lagrangian correspondence is a Lagrangian submanifold $L \subset (M_1 \times M_2, -\omega_1 \oplus \omega_2)$. These are the central objects of the theory of holomorphic quilts as developed by Wehrheim and Woodward in [18]. Consider two Lagrangian correspondences $L_i \subset (M_{i-1} \times M_i, -\omega_{i-1} \oplus \omega_i)$ for $i = 1, 2$. Let

$$\Delta = \{(x, y, z, t) \in M_0 \times M_1 \times M_1 \times M_2 \mid y = z\}$$

If $L_1 \times L_2$ is transverse to Δ, we may form the fibre product $L_1 \circ L_2 \subset M_0 \times M_1 \times M_1 \times M_2$ by intersecting Δ with $L_1 \times L_2$. If the projection $L_1 \circ L_2 \to M_0 \times M_2$ is an embedding, $L_1 \circ L_2$ is naturally a Lagrangian submanifold of $M_0 \times M_2$ and is called the geometric composition of L_1 and L_2. As a point set one has

$$L_1 \circ L_2 = \{\, (x, z) \in M_0 \times M_2 \mid \ \exists y \in M_1 \ \text{ such that } \ (x, y) \in L_1 \ \text{ and } \ (y, z) \in L_2\}$$

1.2 Floer cohomology of a cyclic set of Lagrangian correspondences

A cyclic set of Lagrangian correspondences of length k is a set of Lagrangian correspondences $L_i \subset M_{i-1} \times M_i$ for $i = 1, \ldots, k$ such that $(M_0, \omega_0) = (M_k, \omega_k)$.

Given a cyclic set of Lagrangian correspondences, Wehrheim and Woodward in [18] define a Floer cohomology group $HF(L_1, \dots, L_k)$ (see Section 2.1 below for a review). This can be identified with the Floer homology group of the Lagrangians

$$L_{(0)} = L_1 \times L_3 \times \dots \times L_{k-1} \text{ and } L_{(1)} = L_2 \times L_4 \times \dots \times L_k$$

in the product manifold $\underline{\mathrm{M}} = M_0^- \times M_1 \times M_2^- \times \dots \times M_{k-1}$ if k is even. If k is odd, one inserts the diagonal $\Delta_{M_0} \subset M_0^- \times M_0 = M_{k+1}^- \times M_0$ to get a cyclic set of Lagrangian correspondences with even length. (We denote by M^- the symplectic manifold $(M, -\omega)$ where ω is the given symplectic form on M). Given some assumptions on the underlying Lagrangains, one expects an isomorphism

$$HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \simeq HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

when L_r and L_{r+1} are composable. The main goal of the present work is to prove such an isomorphism under a rather general set of assumptions. For instance, let us discuss this isomorphism in the aspherical case. For this we need introduce some notation. Namely, given two Lagrangians $L, L' \subset (M, \omega)$, we consider the path space:

$$\mathcal{P} = \mathcal{P}(L, L') = \{\gamma : [0,1] \to M |\ \gamma(0) \in L\ ,\ \gamma(1) \in L'\}$$

Now pick $x_0 \in L \cap L'$ to be the constant path on a fixed component of $\mathcal{P}$. Then given any path $\gamma \in \mathcal{P}$ in the same component, we can pick a smooth homotopy γ_t such that $\gamma_0 = x_0$ and $\gamma_1 = \gamma$. Then consider the action functional :

$$\mathcal{A} : \mathcal{P}(L, L') \to \mathbb{R}$$
$$\gamma \to \int_{[0,1]^2} \gamma_t^* \omega$$

This is not always well-defined, because in general it depends on the choice of the homotopy γ_t. However, under various topological assumptions, it is possible to avoid this dependence.

A simple case of the main result in this paper is the following statement:

Theorem 1 *Given a cyclic set of compact connected orientable Lagrangian correspondences $L_1, \dots, L_k$ in compact symplectic manifolds $(M_0, \omega_0), \dots, (M_k, \omega_k)$ such that for some r, L_r and L_{r+1} can be composed. Suppose that the following topological properties hold for all $i = 0, \dots, k$:*

(1)
$$\text{If } \int u^*(\omega_i) \geq 0 \text{ for } [u] \in \pi_2(M_i), \text{ then } [u] = 0.$$
$$\text{If } \int u^*(-\omega_i \oplus \omega_{i+1}) \geq 0 \text{ for } [u] \in \pi_2(M_i \times M_{i+1}, L_{i+1}), \text{ then } [u] = 0.$$

(2) *The following action functionals are well-defined.*
$$\mathcal{A} : \mathcal{P}(L_{(0)}, L_{(1)}) \to \mathbb{R}$$
$$\mathcal{A} : \mathcal{P}(L_r \times L_{r+1}, (L_r \circ L_{r+1}) \times \Delta) \to \mathbb{R}$$

Then,

$$HF(L_0, \ldots, L_r, L_{r+1}, \ldots, L_{k-1}) \simeq HF(L_0, \ldots, L_r \circ L_{r+1}, \ldots, L_{k-1})$$

The assumptions (1) are needed to avoid bubbling in various moduli spaces. The assumptions (2) on the other hand are to ensure that the action functional on the relevant path spaces are single valued which ensures that Floer differential squares to zero. These assumptions are already required for the Floer cohomology groups considered above to be well-defined. One could replace them with assumptions of similar nature but not dispose of them altogether.

The analogous result under positive monotonicity assumptions was proved earlier by Wehrheim and Woodward in [18]. The difficulty in extending their proof to our setting is the fact that the strip shrinking argument in [18] might give rise to certain *figure-eight bubbles* for which no removal of singularities is known. Our proof of the theorem above does not involve strip shrinking and does not give rise to figure-eight bubbles. Applying the idea used for the proof of Theorem 1, we will give an alternative proof of the positive monotone case considered in [18].

Theorem 2 *(positively monotone case) Given a cyclic set of compact orientable Lagrangian correspondences $L_1, \ldots, L_k$ in compact connected symplectic manifolds $(M_0, \omega_0), \ldots, (M_k, \omega_k)$ such that for some r, L_r and L_{r+1} can be composed. Let $\tau \geq 0$ be a fixed real number. Suppose that the following topological properties hold:*

$$\begin{aligned} &\text{For any } v : S^1 \times [0,1] \to (\underline{M}; L_{(0)}, L_{(1)}) \\ &\int v^*\omega_{\underline{M}} = \tau I_{Maslov}(v^*L_{(0)}, v^*L_{(1)}) \\ &\text{The minimal Maslov index for disks in } \pi_2(\underline{M}, L_{(0)}) \text{ and } \pi_2(\underline{M}, L_{(1)}) \text{ is } \geq 3. \end{aligned} \tag{3}$$

Then,

$$HF(L_0, \ldots, L_r, L_{r+1}, \ldots, L_{k-1}) \simeq HF(L_0, \ldots, L_r \circ L_{r+1}, \ldots, L_{k-1})$$

Note that we only require monotonicity for the annuli with boundary on $L_{(0)}$ and $L_{(1)}$ which makes the group on the left well-defined. However, it is easy to see that the corresponding monotonicity relation for the group on the right hand side follows from this. Furthermore, the natural map from $\pi_2(\underline{\mathrm{M}}) \to \pi_2(\underline{\mathrm{M}}; L_{(0)}, L_{(1)})$ implies the following monotonicity of the symplectic manifolds M_i, which determines the monotonicity constant τ.

$$[\omega_{M_i}] = \tau c_1(TM_i) \text{ for all } i.$$

Note also that when $\tau = 0$, the symplectic manifolds are exact and necessarily non-compact. In this case, one needs to assume convexity properties at infinity, as for example in [16]. Indeed, the proof is simpler in the exact case and actually the hypothesis of Theorem 1 are satisfied, hence this case is covered by the previous argument.

Finally, we extend the argument to the (strongly) negatively monotone case which is needed for our application.

Theorem 3 *(strongly negative monotone case) Given a cyclic set of compact connected orientable Lagrangian correspondences $L_1, \dots, L_k$ in compact connected symplectic manifolds $(M_0, \omega_0), \dots, (M_k, \omega_k)$ such that for some r, L_r and L_{r+1} can be composed. Let $\tau < 0$ be a fixed real number. Denote $\dim M_i = 2m_i$. Suppose that the following topological properties hold for all $i = 0, \dots, k$:*

$$\begin{gathered}\text{For any } \; v : S^1 \times [0,1] \to (\underline{M}; L_{(0)}, L_{(1)}) \\ \int v^*\omega_{\underline{M}} = \tau I_{Maslov}(v^*L_{(0)}, v^*L_{(1)})\end{gathered} \tag{4}$$

$$\begin{aligned}&\text{If } \int u^*(\omega_i) > 0 \text{ for } [u] \in \pi_2(M_i), \text{ then } \langle [u], c_1(TM_i)\rangle < -m_i + 2. \\ &\text{If } \int u^*(-\omega_i \oplus \omega_{i+1}) > 0 \text{ for } [u] \in \pi_2(M_i \times M_{i+1}, L_{i+1}), \\ &\qquad \text{then } \mu_{L_{i+1}}([u]) < -(m_i + m_{i+1}) + 1.\end{aligned} \tag{5}$$

Then,

$$HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \simeq HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

In all of the cases, the main idea is to construct a particular homomorphism

$$\Phi : HF(L_0, \dots, L_r, L_{r+1}, \dots, L_{k-1}) \to HF(L_0, \dots, L_r \circ L_{r+1}, \dots, L_{k-1})$$

Once Φ is constructed, a simple energy argument shows that Φ is an isomorphism.

The main motivation for proving Theorem 3 is an application to an explicit example. Namely, we apply Theorem 3 to get rid of a technical assumption in the proof of an isomorphism between Lagrangian matching invariants and Heegaard Floer homology of 3-manifolds, which appeared in a previous work of the first author [8].

In order to avoid repetition, we will not give all the details involved in the definition of holomorphic quilts and Floer cohomology of a cyclic set of Lagrangian correspondences. The more comprehensive discussion of foundations of this theory is available in [18].

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2 Morphisms between Floer cohomology of Lagrangian correspondences

2.1 Chain complex of a cyclic set of Lagrangian correspondences

Let us recall that the chain complex $CF(\underline{L})$ associated with $\underline{L} = (L_1, \ldots, L_k)$ is the freely generated group over a base ring Λ on the generalized intersection points $\mathcal{I}(\underline{L})$ where

$$\mathcal{I}(\underline{L}) = \{\underline{x} = (x_1, \ldots, x_k) \mid (x_k, x_1) \in L_1, (x_1, x_2) \in L_2, \ldots, (x_{k-1}, x_k) \in L_k\}$$

The role of Λ here is no different than its role in the usual Lagrangian Floer cohomology. We will mostly take Λ to be $\mathbb{Z}_2$ (or more generally Novikov rings over a base ring of characteristic 2) in order to avoid getting into sign considerations. The full discussion of orientations in this set-up appeared in [20], from which one expects that under assumptions on orientability of the relevant moduli spaces (say when L_i are relatively spin), our results still hold over $\mathbb{Z}$.

By perturbing $\underline{L}$ with a suitable Hamiltonian isotopy on each M_i, one ensures that $\mathcal{I}(\underline{L})$ is a finite set. More precisely, fix $(\delta_i > 0)_{i=1,\ldots,k}$ and consider the path space:

$$\mathcal{P}(\underline{L}) = \{(\gamma_0, \ldots, \gamma_k) | \ \gamma_i : [0, \delta_i] \to M_i, (\gamma_i(\delta_i), \gamma_{i+1}(0)) \in L_{i+1})\}$$

Now, given Hamiltonian functions $H_i : [0, \delta_i] \times M_i \to \mathbb{R}$, we consider the perturbed gradient flow lines, which form a set of generators of the chain complex $CF(\underline{L})$:

$$\mathcal{I}(\underline{L}) = \{(\gamma_1, \ldots, \gamma_k) | \ \gamma_i'(t) = X_{H_i}(\gamma_i(t)), (\gamma_i(\delta_i), \gamma_{i+1}(0)) \in L_{i+1}\}$$

In the case that $H_i \equiv 0$ for all i, this set coincides with the previous definition given above. It is an easy lemma to show that for a generic choice of $(H_i)_{i=1,\ldots,k}$, the set $\mathcal{I}(\underline{L})$ is finite (see [19] page 7).

Next, to define the differential on $CF(\underline{L})$ we choose compatible almost complex structures $(J_i)_{i=1,\ldots,k}$ on $(M_i)_{i=1,\ldots,k}$ and extend the definition of the Floer differential to our setting in the following way. Let $\underline{x}, \underline{y}$ be generalized intersection points in $\mathcal{I}(\underline{L})$. We define the moduli space of finite energy quilted holomorphic strips connecting $\underline{x}$ and $\underline{y}$ by

$$\begin{aligned}\mathcal{M}(\underline{x}, \underline{y}) = \{u_i : \mathbb{R} \times [0, \delta_i] \to M_i | \ \bar{\partial}_{J_i, H_i} u_i = \partial_s u_i + J_i(\partial_t u_j - X_{H_i}(u_i)) = 0,\\ E(u_i) = \int u_i^* \omega_i - d(H_i(u_i)) dt < \infty\\ \lim_{s \to -\infty} u_i(s, \cdot) = x_i, \ \lim_{s \to +\infty} u_i(s, \cdot) = y_i\\ (u_i(s, \delta_i), u_{i+1}(s, 0)) \in L_{i+1} \text{ for all } i = 1, \ldots k\}/\mathbb{R}\end{aligned}$$

Under certain monotonicity assumptions, it is proven in [18] that, given $(\delta_i)_{i=1,\dots,k}$ and $(H_i)_{i=1,\dots,k}$, there is a Baire second category subset of almost complex structures $(J_i)_{i=1,\dots,i=k}$ for which these moduli spaces are cut out transversely and compactness properties of the usual Floer differential carry over. It is straightforward to check that the same result holds when we replace the monotonicity assumptions by the set of assumptions in the statement of Theorem 1 (for more details, see the proof of Theorem 5.2.3 in [18]). As we check in the proof of Theorem 3, the assumptions of Theorem 3 also gives rise to well-defined moduli spaces. Therefore, in either case one can define the Floer differential for a cyclic set of Lagrangian correspondences by :

$$\partial \underline{\mathrm{x}} = \sum_{\underline{\mathrm{y}} \in \mathcal{I}(\underline{L})} \#\mathcal{M}(\underline{\mathrm{x}}, \underline{\mathrm{y}})\underline{\mathrm{y}}$$

where # means counting isolated points with appropriate sign. Note that if one uses a Novikov ring as the base ring Λ, then the above differential should be modified accordingly as usual to accommodate various other quantities of interest (homotopy class, area,. . . etc.).

The compactness and gluing properties of the above moduli spaces allow one to prove that the differential squares to zero, hence we get a well-defined Floer cohomology group. We refer the reader to Proposition 5.3.1 in [18] for a continuation argument which shows that the resulting group is independent of the choices of $(\delta_i, H_i, J_i)_{i=1,\dots,k}$.

Following [19], we will prove Theorems 1,2 and 3 in a special case (the general case is proved in exactly the same way). Let $(M_i, \omega_i)_{i=0,1,2}$ be symplectic manifolds of dimension $2n_i$ and let

$$L_0 \subset M_0,\ L_{01} \subset M_0^- \times M_1,\ L_{12} \subset M_1^- \times M_2,\ L_2 \subset M_2^-$$

be compact Lagrangian submanifolds such that the geometric composition $L_{02} = L_{01} \circ L_{12} \subset M_0^- \times M_2$ is embedded. As discussed above, we can perturb L_0 and L_2 so that the generalized intersections of $(L_0, L_{01}, L_{12}, L_2)$ as well as (L_0, L_{02}, L_2) are transverse. Our goal is to construct a map

$$\Phi : HF(L_0, L_{01}, L_{12}, L_2) \to HF(L_0, L_{02}, L_2)$$

which we will prove to be an isomorphism. Note that there is an obvious bijection of the chain groups

$$CF(L_0, L_{01}, L_{12}, L_2) \cong CF(L_0, L_{02}, L_2)$$

The map Φ will not necessarily be induced by this bijection. As we will later demonstrate, it will differ from this bijection by a nilpotent matrix.

2.2 Defining the quilt

Our construction of Φ is summarized in Figure 1 below.

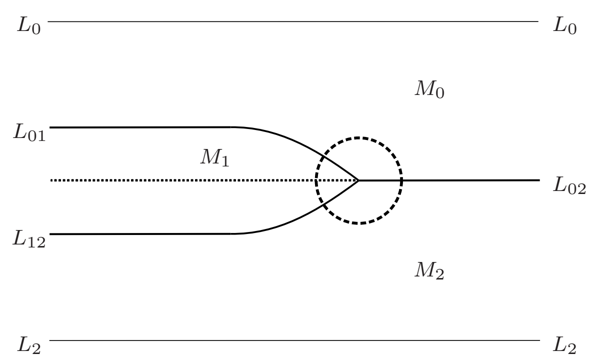

Figure 1: The quilt

Let Σ be the pictured *quilt*. More precisely, ignoring the dotted lines for the moment, on each patch Σ_i we fix a complex structure j_i with real analytic boundary conditions as in [17] (In short, this means that the "seams" are embedded as real analytic sets in Σ). As in Figure 1, let us label the maps from the three patches as $u_i : \Sigma_i \to M_i$ with $i = 0, 1, 2$. At the incoming and outgoing ends, these have the labeled "seam conditions" as in the picture. This means that there are choices of diffeomorphisms between adjacent boundary components of each patch such that the two adjacent maps at a seam can be considered as a map to the product manifolds and that the values of this map lie in the labeled Lagrangian submanifold in the product. (For the interested reader, we refer to Definition 3.1 in [17] for a precise definition of a holomorphic quilt). Inside the dotted circle, we may identify the "Y-end" with $[0, \infty) \times [0, 1]$ mapping to $M_0^- \times M_1 \times M_1^- \times M_2$. More precisely, first we split the strip corresponding to $u_1 : \Sigma_1 \to M_1$ along the dotted horizontal seam in Figure 1 and put the diagonal seam condition $\Delta \subset M_1 \times M_1^-$. This has no effect on the moduli space that we consider. However, now at the Y-end we can "fold" the strip to get the desired map. Specifically, at the Y-end instead of looking at maps from different strips to different manifolds, one can consider a single map from $[0, \infty) \times [0, 1]$ to the product $M_0^- \times M_1 \times M_1^- \times M_2$. Therefore, we choose our complex structure j_i so that near the Y-end they are identified with the standard complex structure on $[0, \infty) \times [0, 1]$. Similarly, we can choose j_i near the incoming and outgoing ends so that we can identify our strips with $(-\infty, 0] \times [0, 1]$ and $[0, \infty) \times [0, 1]$. At the Y-end, let us label the map obtained by folding by

$$v : [0, \infty) \times [0, 1] \to M_0^- \times M_1 \times M_1^- \times M_2 = \underline{\mathrm{M}}$$

This has the seam conditions $v(s, 0) \in L_{01} \times L_{12}$ and $v(s, 1) \in L_{02} \times \Delta$.

Next, we would like to specify the complex structures on each M_i. Assume that we have chosen J_i on M_i such that $HF(L_0, L_{01}, L_{12}, L_2)$ and $HF(L_0, L_{02}, L_2)$ are both defined. In general, such J_i may

need to be t-dependent near

$$(s,t) \in [0,\infty) \times [0,1] \cup (-\infty,0] \times [0,1]$$

to ensure transversality for the moduli spaces that appear in the definition of Floer differential. Note that this specifies $\underline{J} = J_0 \times -J_1 \times J_1 \times -J_2$ on $M_0^- \times M_1 \times M_1^- \times M_2$. To ensure transversality for the moduli space of quilted maps, we now introduce a domain dependent $J(z)$ on M. Pick a small holomorphically embedded disk $D \subset (0,\infty) \times (0,1)$ (Note that this is an interior disk). We define $J(z)$ by letting $J(z) = \underline{J}$ outside D and letting $J(z)$ be chosen generically from the set of compatible complex structures inside D. Such a $J(z)$ need not preserve the product structure on $\underline{M}$. A similar construction in quilted Floer theory already appears in [15]. We will call J the domain dependent complex structure constructed for our quilt.

Definition 4 *Let $\underline{x}, \underline{y}$ be two generalized intersection points for (L_0, L_{02}, L_2) (or equivalently $(L_0, L_{01}, L_{12}, L_2)$). Let $\mathcal{M}_J(\underline{x}, \underline{y})$ be the set of all finite energy maps $\underline{u} = (u_i)_{i=0}^2$ that are holomorphic with respect to J, have the quilted Lagrangian boundary conditions and converge to $\underline{x}$ on the incoming end and to $\underline{y}$ on the outgoing end.*

Note that $L_{02} \times \Delta$ and $L_{01} \times L_{12}$ intersect cleanly in

$$\tilde{L}_{02} = (L_{02} \times \Delta) \cap (L_{01} \times L_{12})$$

which is diffeomorphic to L_{02}. By definition, this means that

$$T\tilde{L}_{02} = T(L_{02} \times \Delta) \cap T(L_{01} \times L_{12})$$

The finite energy assumption guarantees that the map near the Y-end has exponential decay. More precisely, at the Y-end we have a holomorphic map

$$v : [0,\infty) \times [0,1] \to M_0^- \times M_1 \times M_1^- \times M_2$$

for which we have the following decay estimate (see [21], lemma 2.6 or [5], appendix 3):

Lemma 5 *There exists $\epsilon_0 > 0$, such that for any holomorphic v with finite energy there exists C such that*

$$sup_{t\in[0,1]}|\nabla v(s,t)| \leq Ce^{-\epsilon_0 s}$$

This lemma combined with Gromov compactness, implies that each v converges exponentially fast to a point $z \in \tilde{L}_{02}$ as $s \to \infty$. This convergence holds for any C^k-norm on v for $k \geq 0$. We will denote this point by $v(\infty)$.

2.3 Morse-Bott intersections and transversality

Given that any element $u \in \mathcal{M}_J(\underline{x}, \underline{y})$ has exponential decay for some uniform $\epsilon_0 > 0$ at the Y-end, we can view $\mathcal{M}_J(\underline{x}, \underline{y})$ as the zero set of a Fredholm bundle with exponential weights at the Y-end. We briefly review this construction with the purpose of identifying the relevant tangent spaces.

Fix some $p > 2$, critical points $x \in CF(L_0, L_{01}, L_{12}, L_2)$ and $y \in CF(L_0, L_{02}, L_2)$. Let $[0, \infty) \times [0, 1]$ be a neighborhood of the Y-end in the quilt Σ. Let $\overline{\Sigma} = \Sigma - (1, \infty) \times [0, 1]$ be the complement of a slightly smaller end. On $\overline{\Sigma}$ we define the Banach manifold B_1 of all $L^p_{1,\epsilon}$ maps with quilted boundary conditions that converge to x on the incoming end and to y on the outgoing end. For any sufficiently small $\epsilon > 0$, we may define on $[0, \infty) \times [0, 1]$ the Banach manifold B_2 of all $L^p_{1,\epsilon}$ maps

$$v : [0, \infty) \times [0, 1] \to M_0 \times M_1 \times M_1 \times M_2 = \underline{\mathrm{M}}$$

with Lagrangian boundary conditions $v(s, 0) \in L_{01} \times L_{12}$, $v(s, 1) \in L_{02} \times \Delta$ and exponential decay with coefficient ϵ. For a recent review of exponential weights (with references to older treatments) see [21]. Any element $v \in B_2$ converges to some point on the manifold $(L_{01} \times L_{12}) \cap (L_{02} \times \Delta)$. The tangent space to such v are a pair $v' = (v'_0, v'_1)$ where v'_0 is a section of $v^*(T\underline{\mathrm{M}})$ with totally real boundary conditions and v'_1 is just an element of the finite dimensional space $T_{v(\infty)}((L_{01} \times L_{12}) \cap (L_{02} \times \Delta))$. On the level of tangent spaces, the requirement that v' has exponential decay is the statement that

$$\int_{[0,\infty)\times[0,1]} |e^{\epsilon s} v'_0|^p + |\nabla(e^{\epsilon s} v'_0)|^p ds dt < \infty$$

A chart for B_2 near v may be obtained as follows. First, as in [21], we may alternatively view $v' = w_0 + w_1$ where $w_i \in v^*(T\underline{\mathrm{M}})$ with totally real boundary conditions. Here we require that w_0 has exponential decay as $s \to \infty$ and w_1 is covariantly constant near the infinite end. A chart of B_2 near v is obtained by applying the exponential map to all such $v' = w_0 + w_1$ where the norm of each w_i is sufficiently small. To be precise, one must use a t-dependent metric on $\underline{\mathrm{M}}$ which makes $L_{01} \times L_{12}$ totally geodesic for $t = 0$ and $L_{02} \times \Delta$ totally geodesic for $t = 1$. See [21] for more details. Finally, we define our Banach manifold $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$ over Σ as pairs $(a, b) \subset (B_1, B_2)$ which agree on the overlap $[0, 1] \times [0, 1]$.

Now, let $\mathcal{V}$ be the Banach bundle over $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$ whose fibre over $\underline{u}$ is given by $\Omega^{0,1}(\Sigma, \underline{E})$ where $\underline{E}$ is the pullback of the tangent bundles of M_i and $\Omega^{0,1}(\Sigma, \underline{E})$ denotes the space of $(0, 1)$-forms with finite L^p norm and with exponential decay at the Y-end. On each of the two pieces, standard arguments (see [17] for B_1 and [21] for B_2) imply that the $\bar{\partial}$ operator is a restriction of a Fredholm operator to an open domain. A standard patching argument (see for example [2]) implies that the $\bar{\partial}$ operator defines a Fredholm section over $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$. Note that ϵ has to be chosen sufficiently small to ensure that $\bar{\partial}$ is Fredholm on B_1 and that each element of $\mathcal{M}_J(\underline{x}, \underline{y})$ actually belongs to $\mathcal{B}_\epsilon(\underline{x}, \underline{y})$.

For future reference, note that the linearization of $\bar{\partial}$ at some v on $[0,\infty)\times[0,1]$ has the form:

$$D\bar{\partial}\oplus K : L^p_{1,\epsilon}(v^*(T\underline{\mathrm{M}}))\oplus T_{v(\infty)}((L_{01}\times L_{12})\cap(L_{02}\times\Delta))\to L^p_{\epsilon}(v^*(T\underline{\mathrm{M}})\otimes\Omega^{0,1}(\underline{\mathrm{M}}))$$

where K is some operator with a finite dimensional linear domain $T_{v(\infty)}((L_{01}\times L_{12})\cap(L_{02}\times\Delta))$. The specific form of K depends on the choice of the t-dependent metric g_t and will not need to be made explicit for our purposes. The special case when v is constant will be discussed below. We now give a proof of the following claim:

Proposition 6 *For a generic choice of J, $\mathcal{M}_J(\underline{x},\underline{y})$ is a smooth finite dimensional manifold.*

Proof We need to verify that for a generic choice of J this section will be transverse to the zero section. Let us denote $\mathcal{J}$ be the space of almost complex structures constructed above. We will consider domain dependent almost complex structures where this dependence is at the Y-end and only on a small disk $D\in(0,\infty)\times(0,1)$. Thus, given that $u_i\in\mathcal{M}_J(x,y)$, we need to show that the linearized operator $D\bar{\partial}(\underline{\mathrm{u}},J)$ is surjective. Note that as we use a domain dependent the linearized operator has two pieces coming from $D\bar{\partial}(\underline{\mathrm{u}})$ corresponding to variations of $\underline{\mathrm{u}}$ and the second piece corresponds to variations of J. First assume that the map v defined at the Y-end is non-constant. We will show that any section η orthogonal to the linearization must vanish around some point in D. Then unique continuation principle will yield that η vanishes identically.

We now write the linearization of our section on the disk $D\subset(0,\infty)\times(0,1)$, where we required J to have the domain dependence. The main point here is that since we allow our J to be domain dependent, we do not need a somewhere injective curve but simply a point $z_0\in D$ such that $dv(z_0)\neq 0$. Following the argument in [12, page 48], the linearized operator has the following form on D:

$$D\bar{\partial}(v,J)=D\bar{\partial}(v)+\frac{1}{2}S(v,z)\circ dv\circ j_D$$

Here, $D\bar{\partial}(v)$ denotes the differential holding J fixed and $S(z,v)\circ v\circ j_D$ corresponds to linearization with respect to J, where $S(z,v)$ is a section of the tangent space to $\mathcal{J}$ at J which can be identified with $End(TM,J,\omega)$ with

$$SJ=-JS,\ \ \omega(S\cdot,\cdot)=-\omega(\cdot,S\cdot)$$

Now, suppose that some section η is orthogonal to the image. Following [12], we can choose $S(z_0,v(z_0))$ such that

$$\langle\eta(z_0),S(z_0,v(z_0))\circ dv(z_0)\circ j_D\rangle>0$$

whenever $\eta(z_0)\neq 0$. We can extend $S(z_0,v(z_0))$ to a small neighborhood of z_0 by using a bump function. Note that the resulting $S(z,v)$ is domain dependent. This shows that $\eta(z_0)=0$ for all z_0

where $dv(z_0) \neq 0$. However, such z_0 are dense in D. Since η is in the kernel of a $\bar{\partial}$-type operator, namely $(D\bar{\partial}(v))^*\eta = 0$, it must vanish everywhere by the unique continuation principle.

Finally, assume that v is constant, thus by unique continuation $\underline{u}$ is constant. By the index calculation in the next subsection, we know that the index of the linearization is 0. We claim that for any compatible choice J, $D\bar{\partial}(\underline{u}, J)$ is surjective. In view of the index calculation of the next subsection, it is enough to show that the kernel of the linearization at a constant map is zero. We may identify the image of u_i with $0 \in \mathbb{R}^{2n_i}$. The kernel of the linearization is then a triple of maps u_i' from the quilt to $(\mathbb{R}^{2n_i}, \omega_0)$ which satisfy $\bar{\partial}_J u_i', = 0$ In addition, u_i' have linear quilted Lagrangian boundary conditions. Thus, along each seam, $(u_i', u_{i+1}') \in L_{i,i+1}' \subset \mathbb{R}^{2(n_i+n_{i+1})}$ where $L_{i,i+1}'$ is a linear Lagrangian submanifold. Note that by assumption J is compatible with the symplectic structure on $(\mathbb{R}^{2n_0}, -\omega_0) \times (\mathbb{R}^{2n_1}, \omega_0) \times (\mathbb{R}^{2n_1}, -\omega_0) \times (\mathbb{R}^{2n_2}, \omega_0)$ By construction, u_i' have exponential decay near the incoming and outgoing ends. Near the Y-end, we can identify u_i' with the folded map v'. We have

$$v' = v_0' + v_1'$$

where v_0' has exponential decay and v_1' is constant near infinity. We claim that in fact each u_i' is constant. Since u_i' are in the kernel of the linearization, they are holomorphic with respect to J, therefore we have (see [12, page 20]):

$$\frac{1}{2}\int_{\Sigma_i} |du_i'|_J^2 dvol_{\Sigma_i} = \int_{\Sigma_i} (u_i')^*\omega_0$$

If we put $\omega_0 = \frac{1}{2}d(xdy - ydx)$ with $J_{st}(\partial_{x_i}) = \partial_{y_i}$ for the standard complex structure J_{st} on each $\mathbb{R}^{2n_i}$, we get

$$\int_{\Sigma_i} (u_i')^*\omega_0 = \frac{1}{2}\int_{\partial\Sigma_i} \left\{(u_i')_x \partial_s (u_i')_y - (u_i')_y \partial_s (u_i')_x\right\} ds = -\frac{1}{2}\int_{\partial\Sigma_i} \langle u_i', J_{st}\partial_s u'{}_i\rangle ds$$

Here the inner product is the standard inner product, s parametrizes $\partial\Sigma_i$ and $(u_i')_x$, $(u_i')_y$ stand for the x and y components of u_i'. Since the Lagrangian boundary conditions are linear Lagrangian subspaces for the standard symplectic form we have

$$\sum_i \langle u_i', J_{st}\partial_s u'{}_i\rangle = 0$$

and therefore

$$\sum_i \int_{\Sigma_i} |du_i'|_J^2 = -\sum_i \int_{\partial\Sigma_i} \langle u_i', J_{st}\partial_s u'{}_i\rangle = 0$$

Thus, u_i' are constant. Since u_i' converge to zero at the incoming and outgoing ends, this implies that $u_i' = v' = 0$ as desired. □

2.4 Maslov index

In this section we intend to calculate the Maslov index of the linearization at a constant map. Therefore, the linearized problem we will study is that of holomorphic quilts mapping into $\mathbb{C}^n$ with linear Lagrangian boundary conditions. As preparation for the main result, we first review a standard Morse-Bott index calculation. Let L, $L' \subset \mathbb{C}^n$ be a pair of Lagrangian subspaces. Let $\Sigma = \mathbb{R} \times [0,1]$. We consider the Fredholm map

$$\bar{\partial} : L^2_{1,\epsilon}(\Sigma; L, L') \to L^2_{\epsilon}(\Sigma)$$

for sufficiently small ϵ. Here $L^2_{1,\epsilon}(\Sigma; L, L')$ denotes the weighted Sobolev space of maps with $u(\cdot, 0) \in L$ and $u(\cdot, 1) \in L'$.

Lemma 7 *$ind(\bar{\partial}) = -dim(L \cap L')$.*

Proof A function $u : \mathbb{R} \times [0,1] \to (\mathbb{C}^n; L, L')$ may be written as

$$u(s,t) = \sum_{\lambda} f(t)\phi_\lambda(s)$$

where ϕ_λ is an eigenfunction with eigenvalue λ of the operator $-i\partial_s$ on $[0,1]$ with $\phi_\lambda(0) \in L$ and $\phi_\lambda(1) \in L'$.

The kernel of $\bar{\partial} = \partial_t + i\partial_s$ consists of maps

$$u(s,t) = \sum_{\lambda} c_\lambda e^{t\lambda}\phi_\lambda(s)$$

However, since λ is real and such solutions are required to have exponential decay for $t \to \pm\infty$, they must vanish. The cokernel can be identified with the kernel of $-\partial_t + i\partial_s$ on the space of L^2_1 functions with exponential growth of at most ϵ. In addition, these functions have boundary values on iL and iL'. Therefore, such maps consist of

$$u(s,t) = \sum_{\lambda} c_\lambda e^{-t\lambda}\phi_\lambda(s)$$

However, if ϵ is smaller than the first nonzero eigenvalue λ_0, the only maps are those with $\lambda = 0$. These are precisely the constant maps with values in $(iL) \cap (iL')$. Therefore, index $\bar{\partial} = -\mathrm{dim}((iL) \cap (iL')) = -\mathrm{dim}(L \cap L')$. □

This lemma is useful when considering Morse-Bott moduli spaces. In particular, consider the tangent space at the constant map of the moduli space of holomorphic curves with Morse-Bott boundary conditions along (L, L'). By definition, it is the kernel of the map

$$\bar{\partial} \oplus K : L^2_{1,\epsilon}(\Sigma; L, L') \oplus ((L \cap L') \times (L \cap L')) \to L^2_{\epsilon}(\Sigma)$$

For the calculation of the index the explicit form of the map K is not relevant since it is a compact operator. Thus, the index of this linearization is $dim(L \cap L')$. This is consistent with the intuition that the Morse-Bott case corresponds to constant holomorphic disks lying on $L \cap L'$.

We will make use of excision for our index calculations. This is a standard tool for computing the index of elliptic operators that goes back to the work of Atiyah and Singer on the index theorem. We review a simple version of it that is tailored to our application. For recent proofs, one may consult [1].

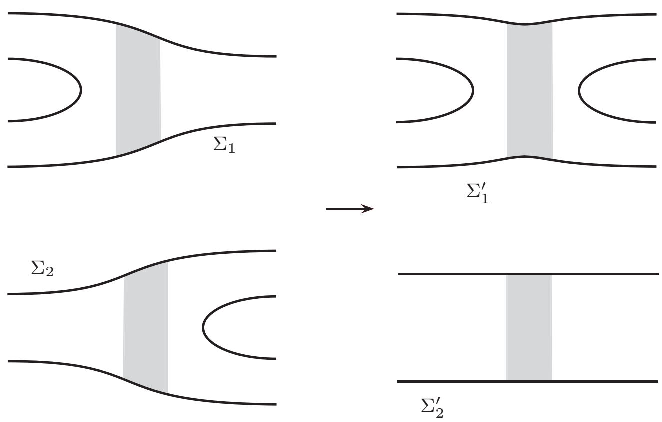

Figure 2: Excision

Suppose we are given quilts Σ_1, Σ_2 each with a pair of complex vector bundles E_i and F_i. In addition, suppose we have $\bar{\partial}$-operators

$$\bar{\partial}_i : \Gamma(E_i) \to \Gamma(F_i)$$

over each Σ_i. At the boundaries, we assume there are totally real boundary conditions. This amounts to a choice of a totally real subbundle T_i of each E_i over the boundary of Σ_i.

Now, assume that each Σ_i contains a separating strip $(a,b) \times [0,1]$. We assume there are isomorphisms $F : E_{1|(a,b)\times[0,1]} \to E_{2|(a,b)\times[0,1]}$ and $G : F_{1|(a,b)\times[0,1]} \to F_{2|(a,b)\times[0,1]}$ which maps $\bar{\partial}_1$ to $\bar{\partial}_2$ and T_1 to T_2. We may excise Σ_i along the strips as in Figure 2 to form new quilts Σ_1' and Σ_2' with corresponding bundles and $\bar{\partial}$-operators $\bar{\partial}_1'$ and $\bar{\partial}_2'$. The excision theorem asserts that

$$\mathrm{ind}(\bar{\partial}_1) + \mathrm{ind}(\bar{\partial}_2) = \mathrm{ind}(\bar{\partial}_1') + \mathrm{ind}(\bar{\partial}_2')$$

A similar discussion applies when instead of a separating strip we have a separating cylinder $(a,b) \times S^1$

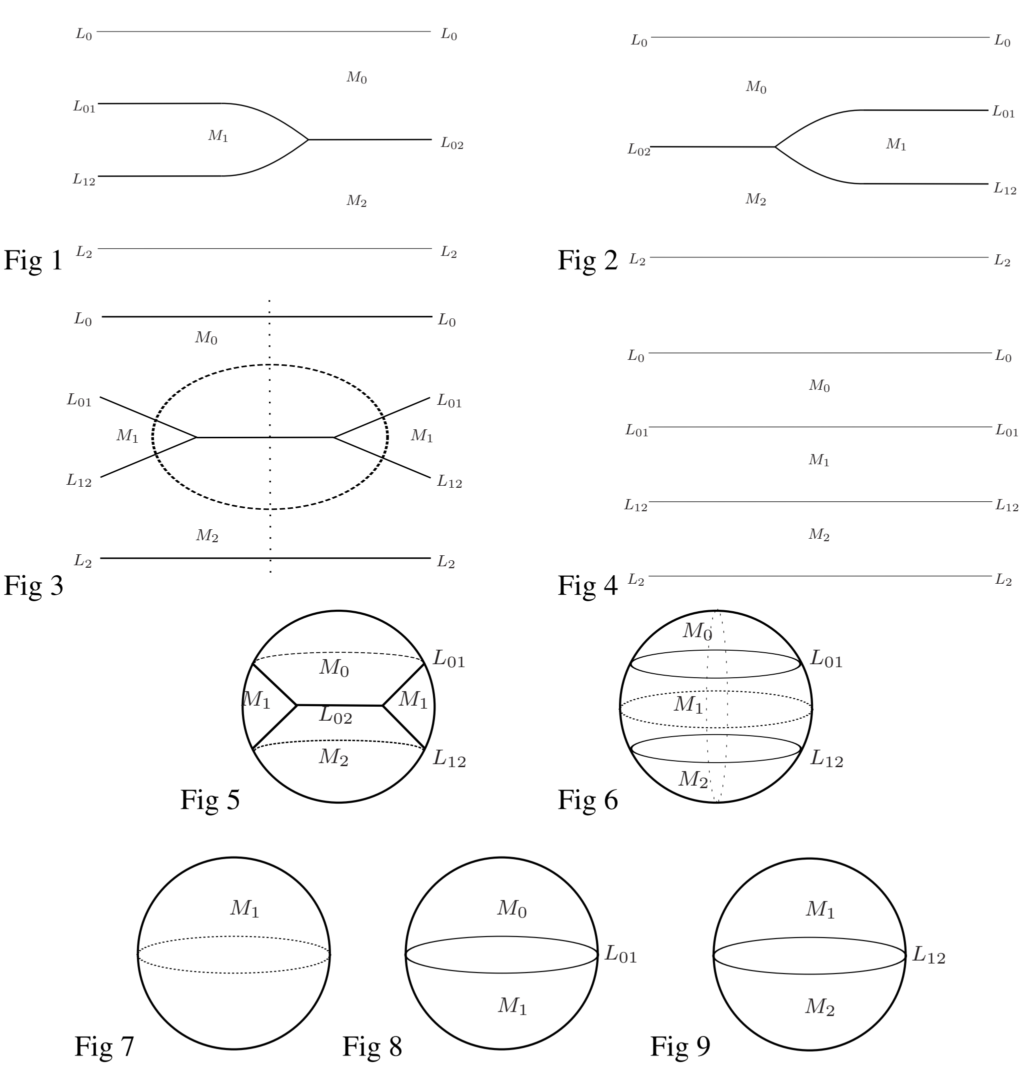

Figure 3: Index calculation using excision

We are ready to compute the index of the linearization at the constant Y-map. Note that for this linearization all maps are into $\mathbb{C}^n$ with the standard complex structure and the nonlinear Lagrangian boundary conditions are replaced by their tangent spaces in $\mathbb{C}^n$. Consider the nine figures drawn in Figure 3. Let m_i stand for the index of Fig i. We wish to compute m_1. We have shown in the previous section that the kernel of the map represented by Fig 1 is zero. Similarly the kernel of Fig 2 is zero. This implies that $m_1 \leq 0$ and $m_2 \leq 0$. By additivity of index,

$$m_3 = m_1 + m_2$$

Excising Fig 3 and 6 along the dotted circles gives

$$m_3 + m_6 = m_4 + m_5$$

Now, we claim that $m_5 = \dim(L_{02})$. To see this, one simply folds to obtain a single strip with Morse-Bott Lagrangian boundary conditions on $(L_{01} \times L_{12}, L_{02} \times \Delta)$. Thus, the discussion right after Lemma 7 above gives $m_5 = \dim(L_{02})$. We have $m_4 = 0$ since it is the identity map. To compute m_6, note that excision implies that

$$m_6 + m_7 = m_8 + m_9$$

By folding, we have that m_8 and m_9 represent disks so

$$m_8 + m_9 = \dim(L_{01}) + \dim(L_{12})$$

and $m_7 = 2\dim(M_1)$ since it is the linearization of a constant map. Thus, $m_6 = \dim(L_{02})$ which together with $m_4 = 0$ and $m_5 = \dim(L_{02})$ gives $m_3 = 0$. This implies $m_1 = m_2 = 0$, as desired.

2.5 Completion of the proof of Theorem 1

First, to define a count we need to show that the zero dimensional moduli space $\mathcal{M}^0_J(\underline{x}, \underline{y})$ is compact and hence finite. Then, $\mathcal{M}^0_J(\underline{x}, \underline{y})$ allows us to define the map

$$\Phi : CF(L_0, L_{01}, L_{12}, L_2) \to CF(L_0, L_{02}, L_2)$$

To verify that this is indeed a chain map we need to consider the 1-dimensional moduli spaces $\mathcal{M}^1_J(\underline{x}, \underline{y})$.

First note that the set of assumptions (1) on second homotopy classes ensures that we cannot have any interior disk or sphere bubbles. Therefore, by Gromov compactness the boundary of the $\mathcal{M}^0_J(\underline{x}, \underline{y})$ and $\mathcal{M}^1_J(\underline{x}, \underline{y})$ consists of broken configurations at the ends. In the case of $\mathcal{M}^0_J(\underline{x}, \underline{y})$, there cannot be breaking at the $\underline{x}$ and $\underline{y}$ ends because by our transversality assumptions such a break will violate the index zero condition. Finally, we need to argue that for both $\mathcal{M}^0_J(\underline{x}, \underline{y})$ and $\mathcal{M}^1_J(\underline{x}, \underline{y})$ there cannot be a breaking at the Y-end.

For this, we first observe that a bubble at the Y-end would be a holomorphic map $u : \mathbb{R} \times [0,1] \to M_0^- \times M_1 \times M_1^- \times M_2$ such that at the end points, this map converges to possibly distinct points, say x_0 and x_1 but they both lie in $\tilde{L}_{02} = (L_{02} \times \Delta) \cap (L_{01} \times L_{12})$. Next, we observe that the assumptions (2) gives a well-defined symplectic action functional on each connected component of the path space $\mathcal{P} = \mathcal{P}(L_{02} \times \Delta, L_{01} \times L_{12}) = \{\gamma : [0,1] \to M_0^- \times M_1 \times M_1^- \times M_2 \,|\, \gamma(0) \in L_{02} \times \Delta \,,\ \gamma(1) \in L_{01} \times L_{12}\}$. Namely, pick $x_0 \in \tilde{L}_{02}$ to be the constant path in the given component of $\mathcal{P}$. Then given any path $\gamma \in \mathcal{P}$ in the same component, we can pick a smooth homotopy γ_t such that $\gamma_0 = x_0$ and $\gamma_1 = \gamma$. Then consider the action functional :

$$\begin{aligned} \mathcal{A} : \mathcal{P}(L_{02} \times \Delta, L_{01} \times L_{12}) &\to \mathbb{R} \\ \gamma &\to \int_{[0,1]^2} \gamma_t^* \omega \end{aligned}$$

The assumptions (2) ensure that $\mathcal{A}$ is well-defined (independent of the chosen homotopy), hence only depends on γ. Note that $\mathcal{A}$ is zero on constant paths (because one can choose γ_t to be a path in $\tilde{L}_{02}$ between x_0 and any constant path.) Now, we observe that a bubble u at the Y-end gives a path in $\mathcal{P}$ connecting the constant paths x_0 and x_1. Since u is a holomorphic map, $\mathcal{A}(x_1)$ is nonzero unless u is constant. So u has to be constant and there cannot be any bubbling at the Y-end.

Therefore, standard gluing theory applied to $\mathcal{M}_J^1(\underline{\mathrm{x}}, \underline{\mathrm{y}})$ shows that Φ is a chain map.

Remark 8 *Note that we do not need to consider Morse-Bott gluing as the only breakings occur at the ends where we have transverse intersection.*

To complete the proof we need to show that Φ induces an isomorphism on cohomology. Let us write $\mathcal{P}^{in} = \{(\gamma_0, \gamma_1, \gamma_2) |\ \gamma_i : [0,1] \to M_i, \gamma_0(0) \in L_0,\ (\gamma_0(1), \gamma_1(0)) \in L_{01},\ (\gamma_1(1), \gamma_2(0)) \in L_{12}, \gamma_2(1) \in L_2\}$

As above, assumptions (2) enable us to have a well-defined action functional,

$$\begin{aligned} \mathcal{A}^{in} : \mathcal{P}^{in} &\to \mathbb{R} \\ \gamma &\to \sum_{i=0}^{2} \int_{[0,1]^2} (\gamma_i^t)^* \omega_i \end{aligned}$$

where as before γ_i^t is any choice of a smooth homotopy in $\mathcal{P}^{in}$ between γ_i and a fixed constant path, and ω_i are the given symplectic forms on M_i. Therefore, the chain complex $CF(L_0, L_{01}, L_{12}, L_2)$ inherits a filtration given by A^{in}. Recall that the Floer differential decreases the action functional.

Next, we have a similar filtration on $CF(L_0, L_{02}, L_2)$, where we write

$$\mathcal{P}^{out} = \{(\gamma_0, \gamma_2) |\ \gamma_i : [0,1] \to M_i, \gamma_0(0) \in L_0,\ (\gamma_0(1), \gamma_2(0)) \in L_{02}, \gamma_2(1) \in L_2\}$$

and $\mathcal{A}^{out}$ is defined as before. Note that $\mathcal{P}^{out}$ consists of elements of $\mathcal{P}^{in}$ such that γ_1 is constant. Therefore, the action functional $\mathcal{A}^{out} = \mathcal{A}^{in}$ whenever both are defined.

Since constant maps are zero dimensional solutions, we have $\mathcal{M}_J^0(\underline{x}, \underline{x}) = 1$. Therefore, to conclude that Φ is an isomorphism, it suffices to show that Φ is a filtered chain map. For this, it suffices to show that if $\mathcal{M}_J^0(\underline{x}, \underline{y})$ is non-empty, then the following inequality holds :

$$\mathcal{A}^{in}(\underline{x}) \geq \mathcal{A}^{out}(\underline{y})$$

This is now easy to show, since let u be a holomorphic curve in $\mathcal{M}_J^0(\underline{x}, \underline{y})$. Now, u can be considered as a path $(\gamma_i^t)_{i=0}^2$ in the path space $\mathcal{P}^{in}$ such that γ_1^t shrinks to a constant path as we get to the Y-end and stays constant until the outgoing end. Now, since u is a holomorphic map, it strictly decreases the action. This gives the desired inequality: $\mathcal{A}^{in}(\underline{x}) \geq \mathcal{A}^{in}(\underline{y}) = \mathcal{A}^{out}(\underline{y})$, where the last equality holds since $\underline{y} \in \mathcal{P}^{out}$. ◻

3 Extensions of the main theorem

In this section, we discuss the proof of Theorem 1 under positive and strongly negative monotonicity assumptions. This result in the positively monotone case was first proved by Wehrheim and Woodward by different techniques. However, the strongly negative monotone case is new and important for our application.

As a first step, we prove a topological lemma, which will allow us to establish a priori energy bound for pseudoholomorphic curves counted in the moduli space $\mathcal{M}_J(\underline{x}, \underline{y})$ used for defining the map $\Phi : CF(L_0, L_{01}, L_{12}, L_2) \to CF(L_0, L_{02}, L_2)$.

Lemma 9 *Let $\underline{x}, \underline{y} \in CF(L_0, L_{01}, L_{12}, L_2) \simeq CF(L_0, L_{02}, L_2)$ be two generalized intersection points. Let $\mathcal{P}(\underline{x}, \underline{y})$ be the space of maps $(-\infty, \infty) \to \mathcal{P}(L_0, L_{01}, L_{12}, L_2)$ which asymptotically converge to $\underline{x}$ and $\underline{y}$. Similarly, let $\mathcal{B}(\underline{x}, \underline{y})$ be the space of smooth maps that is considered for defining the map $\Phi : CF(L_0, L_{01}, L_{12}, L_2) \to CF(L_0, L_{02}, L_2)$ (see Section 2.2) Then there is a natural inclusion map $\mathcal{B}(\underline{x}, \underline{y}) \hookrightarrow \mathcal{P}(\underline{x}, \underline{y})$ which induces an isomorphism on the connected components:*

$$\pi_0(\mathcal{P}(\underline{x}, \underline{y})) \simeq \pi_0(\mathcal{B}(\underline{x}, \underline{y}))$$

In particular, any homotopy class of a map $\Phi : CF(L_0, L_{01}, L_{12}, L_2) \to CF(L_0, L_{02}, L_2)$ mapping $\underline{x}$ to $\underline{y}$ can be represented as a concatenation of maps $\Phi = u\#c$, where $u \in \mathcal{P}(\underline{x}, \underline{y})$ and $c : CF(L_0, L_{01}, L_{12}, L_2) \to CF(L_0, L_{02}, L_2)$ is the constant map with value $\underline{y}$.

Proof Recall that the space of paths $\mathcal{P}(L_0, L_{01}, L_{12}, L_2) = \{(\gamma_1, \gamma_2, \gamma_3) | \gamma_i : [0,1] \to M_i, \gamma_1(0) \in L_0, (\gamma_1(1), \gamma_2(0)) \in L_{01}, (\gamma_2(1), \gamma_3(0)) \in L_{12}, \gamma_3(1) \in L_2)\}$. We will denote a path $\gamma : (-\infty, \infty) \to \mathcal{P}(L_0, L_{01}, L_{12}, L_2)$ in this path space by $\gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s) \in \mathcal{P}(\underline{x}, \underline{y})$, where $s \in (-\infty, \infty)$. Now the space $\mathcal{B}(\underline{x}, \underline{y})$ can be identified as a subspace of $\mathcal{P}(\underline{x}, \underline{y})$ where $\gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s) \in \mathcal{B}(\underline{x}, \underline{y})$ if and only if γ_2^s is a constant with respect to t for $s \geq 1$. More precisely, any map $\Phi \in \mathcal{B}(\underline{x}, \underline{y})$ can be homotoped to be constant near the Y-end (because of the exponential convergence at the Y-end). Now, the domain of Φ can be thought of as a flat plane, identify it with say $\mathbb{R} \times [0,1]$ (as in Figure 1). We arrange so that the Y-end point corresponds to $(1, 1/2)$. Foliating the domain of the map Φ by vertical lines $L_s = \{s\} \times [0,1]$, we obtain the path of paths $(\gamma_1^s, \gamma_2^s, \gamma_3^s)$ as restrictions of $\Phi|_{L_s}$. Note that γ_2^s does not vary with t for $s \geq 1$.

The desired equivalence of path components can be seen by noting that any path $\gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s) \in \mathcal{P}(\underline{x}, \underline{y})$ is homotopic to a path which is constant for $s > N$ for some sufficiently large N by the requirement of convergence as $s \to \infty$. One can then isotope γ^s so that it is constant for $s \geq \frac{1}{2}$ in both s and t. Thus, we have an inverse map $\pi_0(\mathcal{P}(\underline{x}, \underline{y})) \to \pi_0(\mathcal{B}(\underline{x}, \underline{y}))$ to the map induced by the inclusion map. It is easy to see that this gives the desired isomorphism.

To see the last part of the statement more explicitly, express any map Φ as $(\gamma_1^s, \gamma_2^s, \gamma_3^s)$ as above with γ_i^s constant for $s > N$. Now, consider the homotopy Φ^r where $r \in [1, 2N]$ given by $(\gamma_1^{rs}, \gamma_2^{rs}, \gamma_3^{rs})$. Then Φ^{2N} is a map in $\mathcal{B}(\underline{x}, \underline{y})$, which is constant for $s \geq \frac{1}{2}$, hence it is a concatenation of $u \in \mathcal{P}(\underline{x}, \underline{y})$ and the constant map with value $\underline{y}$ as stated. □

We are now ready to prove the extension of Theorem 1 to the monotone case:

3.1 Proof of Theorem 2

We briefly recall from [18] why the Floer cohomology groups in consideration are well-defined (independent of the choices, invariant under Hamiltonian deformations,etc.). Given $\underline{x}, \underline{y} \in L_{(0)} \cap L_{(1)}$, the monotonicity assumptions guarantee that the energy of index k holomorphic strips $u \in \mathcal{M}^k(\underline{x}, \underline{y})$ is constant. Therefore, by Gromov-Floer compactness it suffices to exclude disk and sphere bubbles. The orientation and monotonicity assumptions ensure that any non-trivial holomorphic disk must have Maslov index at least 2 which excludes disk bubbles in 0 and 1 dimensional moduli spaces, however to have a well-defined Floer cohomology group we also need to avoid disk bubbles in index 2 moduli spaces, hence we require the minimal Maslov index for disks to be at least 3 (the sphere bubbles are handled similarly).

Now, as before we will consider the map

$$\Phi : HF(L_0, L_{01}, L_{12}, L_2) \to HF(L_0, L_{02}, L_2)$$

which is defined by counting solutions in $\mathcal{M}^0_J(\underline{x}, \underline{y})$. Let us first study the compactness property of the moduli space $\mathcal{M}_J(\underline{x}, \underline{y})$ under our assumptions. We first need to establish an area-index relation to have a priori energy bound so that we can apply Gromov-Floer compactness. This follows easily from Lemma 9. Namely, to compute the index of an element $\Phi \in \mathcal{M}_J(\underline{x}, \underline{y})$, we can topologically apply a homotopy as in Lemma 9 so that $\Phi = u\#c$, where u is a map contributing to the differential of the chain complex $CF(L_0, L_{01}, L_{12}, L_2)$ and $c \in \mathcal{M}_J(\underline{y}, \underline{y})$ is the *constant* map at $\underline{y}$. In Section 2.4, we computed the index of c to be equal to zero. (Indeed, this computation is the non-trivial part of the argument that we are giving here). Therefore, by excision,

$$\mathrm{index}(\Phi) = \mathrm{index}(u) + \mathrm{index}(c) = \mathrm{index}(u)$$

Now, by the area-index relation for the moduli space that u belongs to (this follows from the monotonicity assumptions, see [18, Remark 5.2.3]) , the energy of index k holomorphic strips is constant. Since energy of Φ is equal to energy u, it follows that the energy of index k maps in $\mathcal{M}_J(\underline{x}, \underline{y})$ is constant. Hence the Gromov-Floer compactness applies.

In view of the area-index relation for Y-maps, we have energy bounds on all trajectories of index 0 and 1 and thus Gromov-Floer compactification holds. Therefore, the compactification includes broken configurations at the ends, possibly also including the Y-end, and disk and sphere bubbles. However, as before our monotonicity assumptions ensure that disk and sphere bubbles do not arise in the compactification of index 0 and 1 moduli spaces. Recall that, $\mathcal{M}^0_J$ is used to define the map Φ and $\mathcal{M}^1_J$ is used to check that it is a chain map.

We now need to deal with bubbles at the Y-end. Given a sequence of trajectories $u_i \in \mathcal{M}^1_J$ breaking along the Y-end, by Gromov-Floer compactness, we get in the limit a pair (u_∞, δ) where u_∞ is a possibly broken Y map and δ is a holomorphic strip with Morse-Bott boundary conditions along $(L_{01} \times L_{12}, L_{02} \times \Delta)$. Note that δ can also be broken but that does not affect the argument. Now, if δ is non-constant, it will have non-zero energy, therefore we have

$$E(u_\infty) < E(u_i)$$

By the energy-index relation proved above and the orientability assumptions, this implies

$$\mathrm{index}(u_\infty) \leq \mathrm{index}(u_i) - 2 = -1$$

Since the index is additive, there exists at least one unbroken holomorphic piece in u_∞ with negative index. However, since these moduli spaces are cut out transversely, this cannot occur. Therefore, $\mathcal{M}^0_J$ and $\mathcal{M}^1_J$ cannot have any broken configuration with a bubble at the Y-end. This concludes the argument that the map Φ is well-defined.

Now, to check that Φ is an isomorphism, we construct an approximate inverse to Φ. Let

$$\Psi : HF(L_0, L_{02}, L_2) \to HF(L_0, L_{01}, L_{12}, L_2)$$

be the map obtained by counting index 0 holomorphic maps obtained by reversing the Y-map. Arguments identical to those for Φ, show that Ψ is a chain map. We claim that

$$\Psi \circ \Phi = I + K$$

where I is the identity and K is nilpotent. This will prove that Φ is an isomorphism. The diagonal entries of $\Psi \circ \Phi$ are obtained by counting pairs of broken trajectories (u_1, u_2) with u_1 starting at a critical point x and u_2 ending at the same critical point. In addition, u_1 has the same endpoint as the starting point of u_2. By the area-index relation, the only such trajectories of index 0 are the constants. More generally, given a sequence of such broken pairs (u_1, u_2), $(u_3, u_4) \cdots (u_{N-1}, u_N)$ such that the endpoint of u_i is the starting point of u_{i+1} and the starting point of u_1 is the same as the endpoint of u_N we have that the only index zero trajectory is the broken constant one. This follows from the monotonicity assumption. Indeed, since index and area are additive, any such trajectory with nonzero area has positive index.

Let N_0 denote the number of critical points. Consider a broken trajectory, that is a sequence of homolomorphic curves that contribute to $\Psi \circ \Phi$, of index zero with no constant pairs. Any such trajectory with more than $N_0 - 1$ pairs must have a repeated critical point. This is impossible since such a segment has positive index, as we just explained. Now, if $K^k(x)$ is nonzero there must be a broken trajectory of length k connecting x to some critical point y. This trajectory consists of non-constant pairs. This is easily seen by induction. First, $K(x)$ lies in the span of critical points connected to x by a non-constant pair. Suppose that $K^i(x)$ lies in the span of critical points y_j connected to x by a non-constant broken path of pairs of length $i-1$. Any nonzero matrix element $\langle y, K(y_i) \rangle$ gives rise to a critical point y connected to y_i by a non-constant pair. It is thus connected to x by a non-constant path of pairs of length i as desired. Therefore, we may conclude that $K^{N_0} = 0$ since it is contained in the span of elements coming from broken pairs of length N_0. This completes the proof that Φ is an isomorphism. □

3.2 Proof of Theorem 3

In this case, we follow the same steps as in the positively monotone case. The only difference is the way we handle various exclusions of bubbles. Namely, we exclude bubbling by first arranging the transversality for the moduli spaces of *simple* sphere bubbles and *simple* disk bubbles. The strongly negative monotonicity assumptions is the assumption that the expected dimension of these moduli spaces is negative therefore when transversality holds (which can be arranged by choosing the almost complex structure J in the target generically), we guarantee that these moduli spaces are empty. A lemma of McDuff ([12], Proposition 2.51) and the decomposition lemma of Kwon-Oh [6] and Lazzarini ([7]) allows us to lift this to non-simple sphere and disk bubbles. More specifically, the lemma of McDuff states that any pseudoholomorphic sphere factors through a simple

pseudoholomorphic sphere, so the existence of the former one implies the existence of the latter. Similarly, Kwon-Oh and Lazzarini's lemma implies that the existence of any pseudoholomorphic disk ensures the existence of a simple pseudoholomorphic disk. Now, recall that the expected dimension of unparameterized moduli space of spheres in M_i in the homology class $[u]$ is $2(\langle [u], c_1(TM_i)\rangle + m_i - 3)$. As part of the hypothesis, we assumed that this number is negative, in fact we assumed that this number is strictly less than -2, to exclude bubbling in $\mathcal{M}^k_J(\underline{\mathrm{x}}, \underline{\mathrm{y}})$, for $k = 0, 1, 2$. This is required to ensure that the Floer cohomology groups that we are considering are independent of the auxiliary choices. Similarly, to avoid disk bubbles, recall that by the real-analyticity of the seams, any disk bubble in a quilted map can be seen as disk bubble in $M_i \times M_{i+1}$ with boundary on L_{i+1} for some i. The expected dimension for unparameterized simple disks in the homology class u is given by $\mu_{L_{i+1}}([u]) + (m_i + m_{i+1}) - 3$. We assumed that this number is strictly less than -2 to avoid disk bubbles in $\mathcal{M}^k_J(\underline{\mathrm{x}}, \underline{\mathrm{y}})$, for $k = 0, 1, 2$ for the same reason as before.

Therefore, these considerations imply that the Floer cohomology groups are well-defined. Furthermore, the negative monotonicity assumption gives an area-index relation as before, which guarantees a priori energy bound on the moduli space $\mathcal{M}^k_J(\underline{\mathrm{x}}, \underline{\mathrm{y}})$, hence Gromov-Floer compactness applies. Since we excluded the possibility of the sphere and disk bubbled configurations in the compactification of the moduli spaces $\mathcal{M}^0_J$ and $\mathcal{M}^1_J$, to finish off the only remaining issue is to exclude the bubbling at the Y-end. We will follow the notation given in the proof of Theorem 2. We need to exclude non-constant δ bubbles. Recall that since

$$\delta : \mathbb{R} \times [0, 1] \to (\underline{\mathrm{M}}; L_{01} \times L_{12}, L_{02} \times \Delta)$$

is a strip with Lagrangian boundary conditions and at the two ends converges exponentially to points in the Morse-Bott intersection. Note that we can always ensure the transversality of the moduli space of such δ bubbles by choosing our J to be t-dependent near the Y-end (cf. [4]).

Lemma 10 *index*$(\delta) \leq 0$.

Proof We will relate the index of δ to that of a disk in $\underline{\mathrm{M}}$ with boundary on $L_{01} \times L_{12}$. The desired conclusion will then follow from the monotonicity assumptions of Theorem 3. For this it will be convenient to view δ as a quilt of maps

$$\delta_i : \mathbb{R} \times [0, 1] \to M_i, i = 1, 2, 3$$

with cyclic Lagrangian boundary conditions $(L_{01}, L_{12}.L_{02})$. For instance, we have $(\delta_2(1, s), \delta_0(0, s)) \in L_{02}$. Let $\delta_4 : \mathbb{R} \times [0, 1] \to M_1$ be the map with $\delta_4(s, t) = b(s)$, where $b(s)$ is the unique point on M_1 with $(\delta_2(1, s), b(s), b(s), \delta_0(0, s)) \subset L_{01} \times L_{12}$. Note that δ_4 is a smooth map which is not holomorphic but converges exponentially as $|s| \to \infty$. Furthermore, the image of δ_4 is just a path, thus δ_4 has zero area. We have now obtained a new quilt δ' with four patches δ_i and seams $(L_{01}, L_{12}, L_{01}, L_{12})$. Note that $E(\delta) = E(\delta')$. We fold δ' to obtain a map

$$\delta'' : \mathbb{R} \times [0, 1] \to \underline{\mathrm{M}}$$

with boundary on $(L_{01}\times L_{12}, L_{01}\times L_{12})$. Alternatively, we may view this as a map

$$\delta''' : D \to \underline{\mathrm{M}}$$

where D is the unit disk and δ''' has Lagrangian boundary conditions on $L_{01}\times L_{12}$. Note that $\mathrm{index}(\delta''') = \mathrm{index}(\delta'')$. This is a well-known statement but we include an explanation of it here for completeness. First, note that when δ'' is constant we have (see section 2.4)

$$\mathrm{index}(\delta'') = \mathrm{index}(\delta''') = m_0 + 2m_1 + m_2$$

In general, after a homotopy, we can assume that the infinite strip is constant near the ends. By excision near the infinite ends, we have

$$\mathrm{index}(\delta'') + \mathrm{index}(\alpha_1) + \mathrm{index}(\alpha_2) = \mathrm{index}(\delta''') + \mathrm{index}(\beta_1) + \mathrm{index}(\beta_2)$$

where α_i are a constant disks with boundary on $L_{01}\times L_{12}$ and β_i are constant map from a "flask" which is diffeomorphic to a disk with one boundary puncture and has one infinite end. Note that

$$\mathrm{index}(\alpha_1) + \mathrm{index}(\alpha_2) - \mathrm{index}(\beta_1) - \mathrm{index}(\beta_2)$$

does not depend on δ'' or δ''' and thus must equal 0 in view of case when δ'' is constant. Therefore, $\mathrm{index}(\delta'') = \mathrm{index}(\delta''')$ as desired. Finally, note that by the monotonicity assumptions,

$$\mathrm{index}(\delta''') = \mu_{L_{12}\times L_{12}}([\delta''']) + m_0 + 2m_1 + m_2 \leq 0$$

We now relate this conclusion to the index of the original δ. We have,

$$\mathrm{index}(\delta) + 2m_1 = \mathrm{index}(\delta'')$$

To see this note that δ and δ'' have the same Maslov index while the dimension of the Morse-Bott intersection for δ'' is $m_0 + 2m_1 + m_2$ and for δ it is $m_0 + m_2$. Putting this together with the computation of $\mathrm{index}(\delta'')$, we get

$$\mathrm{index}(\delta) + 2m_1 \leq 0$$

We conclude that $\mathrm{index}(\delta) \leq 0$ as desired. □

Since δ is assumed to be non-constant, it cannot have expected dimension zero since translations contribute one dimension to the moduli space. Therefore, $\mathrm{index}(\delta) < 0$. Such δ cannot occur in view of the transversality assumptions. Having excluded bubbling at the Y-end, we argue as in the positively monotone case to conclude that the map Φ gives the desired isomorphism. The crucial point is again to exclude broken non-constant trajectories with the same endpoints. While in the positive monotone case these gave rise to moduli spaces of index greater than zero, under the negative monotone assumptions, the sum of the expected dimension of such trajectories is negative. This means that at least one unbroken trajectory in the sequence has negative index. This violates the transversality assumptions on the Y-map. □

4 Applications

In Theorem 25 of [8], the first author proves an isomorphism of Floer groups associated with (Y,f) where Y is a closed 3–manifold with $b_1 > 0$ and f is a circle valued Morse function with connected fibres and no maxima or minima. Such an f is called a *broken fibration* in [8], which we adopt here. We will denote by $\Sigma_{\max}$ and $\Sigma_{\min}$ two fixed regular fibres which have maximal genus g and minimal genus k among fibres of f. We also write $\mathcal{S}(Y|\Sigma_{\min})$ for the set of Spin^c structures $\mathfrak{s}$ on Y, which satisfy $\langle c_1(\mathfrak{s}), \Sigma_{\min}\rangle = \chi(\Sigma_{\min})$.

In the proof of this isomorphism, an important step is an application of Theorem 1, which was available through the work of Wehrheim and Woodward in [18]. However, in an early version of [8] before Theorem 3 was available, the author needed to assume a technical condition, namely that one needs to have a removal of singularity property for the *figure-eight bubbles*, which is needed in order to extend the proof of Wehrheim and Woodward in [18] to this setting. While this property is yet to be proved (or disproved), Theorem 1, more precisely its extension to the strongly negative case given in Theorem 3 in this paper enables us to drop this technical assumption in Theorem 25 of [8]. Therefore, we finally have the desired form of this theorem as follows:

Theorem 11 *(cf. Theorem 25 of [8]) Suppose that Y admits a broken fibration with $g < 2k$. Then for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{min})$,*

$$QFH'(Y,f;\mathfrak{s},\Lambda) \simeq QFH(Y,f;\mathfrak{s},\Lambda)$$

Once we have Theorem 3 and the results obtained in [8], the proof of this result is a matter of verifying that the Lagrangian correspondences and symplectic manifolds involved in the definition of above groups satisfy hypothesis of Theorem 1. For this purpose, we briefly recall the definitions of the groups $QFH'(Y,f;\mathfrak{s},\Lambda)$ and $QFH(Y,f;\mathfrak{s},\Lambda)$.

We remark that the hypothesis $g < 2k$ is required for the group $QFH(Y,f;\mathfrak{s},\Lambda)$ to be well-defined. Without this assumption, we do not know how to avoid the possibility of disk bubbles. On the other hand, the group $QFH'(Y,f;\mathfrak{s},\Lambda)$ has an alternative cylindrical formulation a la Lipshitz ([11]) which makes it well defined in general. (See [8] for a detailed exposition of this issue). Furthermore, the groups in Theorem 11 can be defined over $\mathbb{Z}$ when $k > 1$. In that case, the isomorphism holds with $\mathbb{Z}$ coefficients.

4.1 Quilted Floer homology of a broken fibration

Given a Riemann surface (Σ,j) and a non-separating embedded curve $L \subset \Sigma$, denote by Σ_L the surface obtained after surgery along L, which is given by removing a tubular neighborhood of L and

gluing in a pair of discs. We also choose an almost complex structure $\bar{j}$ on Σ_L which agrees with j outside a neighborhood of L. This is a canonical construction in the sense that the parameter space is contractible.

Perutz equips $\mathrm{Sym}^n(\Sigma)$ with a Kähler form ω_Σ in the cohomology class $\eta_\Sigma + \lambda\theta_\Sigma$ ([13]). Here, η_Σ is the Poincaré dual to $\{pt\} \times \mathrm{Sym}^{k-1}(\Sigma)$, $\theta_\Sigma - g\eta_\Sigma$ is Poincaré dual to $\sum_{i=1}^{g} \alpha_i \times \beta_i \times \mathrm{Sym}^{n-2}(\Sigma)$ where $\{\alpha_i, \beta_i\}$ is a symplectic basis for $H_1(\Sigma)$, and $\lambda > 0$ is a small positive number.

To such data, Perutz in [13] associates a Lagrangian correspondence V_L in $\mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\Sigma_L)$, for any $n \geq 1$. The symplectic forms ω and ω_L on the two spaces are are Kähler forms in the respective cohomology class $\eta + \lambda\theta$, where $\lambda > 0$ is a common parameter.

The correspondence is obtained as a vanishing cycle for a symplectic degeneration of $\mathrm{Sym}^n(\Sigma)$. One first considers a holomorphic Lefschetz fibration E over D^2 with regular fibre Σ and just one vanishing cycle, isotopic to L; thus L collapses, in the fibre over the origin, to a nodal curve Σ_0. The correspondence V_L arises from the vanishing cycle of the relative Hilbert scheme of n points associated with this Lefschetz fibration. Furthermore, Perutz proves that the correspondences V_L are canonically determined up to Hamiltonian isotopy [13].

Using Perutz's constructions, in [8] the first author defined the group $QFH(Y,f,\Lambda)$. Given a 3–manifold Y and a broken fibration $f : Y \to S^1$, the *quilted Floer homology* of Y, $QFH(Y,f)$ is a quilted Floer homology of the cyclic set of Lagrangian correspondences associated with f and a Spin^c structure on Y obtained by the Perutz's construction above.

More specifically, we will restrict our attention to broken fibrations f which have connected fibres and the genera of the fibres are in decreasing order as one travels clockwise and counter-clockwise from -1 to 1. As before, we denote by $\Sigma_{\max}$ a genus g surface identified with the fibre above -1 and by $\Sigma_{\min}$ a genus k surface identified with the fibre above 1. On $\Sigma_{\max}$, we denote by $\alpha_1, \ldots, \alpha_{g-k}$, the disjoint attaching circles of the two handles corresponding to the critical points of f above the northern semi-circle, and similarly we have disjoint attaching circles $\beta_1, \ldots, \beta_{g-k}$ for the critical points of f above the southern semi-circle.

Let $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ be critical values of f in the northern and southern semicircle respectively. Now, pick points $p_i^{\pm}, q_i^{\pm}$ in a small neighborhood of each critical point p so that the fibre genus increases from p_i^- (resp. q_i^-) to p_i^+ (resp. q_i^+).For $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, let $\nu : S^1 \backslash \mathrm{crit}(f) \to \mathbb{Z}_{\geq 0}$ be the locally constant function defined by $\langle c_1(\mathfrak{s}), [F_s] \rangle = 2\nu(s) + \chi(F_s)$, where $F_s = f^{-1}(s)$. We denote the Lagrangian correspondences obtained by Perutz's construction by $V_{\alpha_i} \subset \mathrm{Sym}^{\nu(p_i^+)}(F_{p_i^+}) \times \mathrm{Sym}^{\nu(p_i^-)}(F_{p_i^-})$ and $V_{\beta_i} \subset \mathrm{Sym}^{\nu(q_i^+)}(F_{q_i^+}) \times \mathrm{Sym}^{\nu(q_i^-)}(F_{q_i^-})$

With this notation, $QFH(Y,f)$ is given as the quilted Floer homology of the Lagrangian correspondences $V_{\alpha_1}, \ldots, V_{\alpha_{g-k}}$ and $V_{\beta_1}, \ldots, V_{\beta_{g-k}}$. Note that we use the complex structures of the

form $\mathrm{Sym}^r(j_s)$ on the symplectic manifolds $\mathrm{Sym}^r(\Sigma)$ where j_s is a generic path of almost complex structures on Σ. The fact that these complex structures achieve transversality is standard, see Proposition A.5 of [11] for an argument in a related set-up. As remarked before, to avoid disk bubbles in this setting, we need to assume $g < 2k$. (Note that in the upcoming work [10], $QFH(Y,f)$ is shown to be independent of the choice of f).

Next, we restrict our attention to Spin^c structures which lie in $\mathcal{S}(Y|\Sigma_{\min})$. The following lemma can be found in Appendix B of [9] (see also [10] for an improved exposition):

Lemma 12 *For $g > k$, $V_{\alpha_1} \circ \ldots \circ V_{\alpha_{g-k}}$ and $V_{\beta_1} \circ \ldots \circ V_{\beta_{g-k}}$ are respectively Hamiltonian isotopic to $\alpha_1 \times \ldots \times \alpha_{g-k}$ and $\beta_1 \times \ldots \times \beta_{g-k}$ in $\mathrm{Sym}^{g-k}(\Sigma)$ equipped with a Kähler form ω which lies in the cohomology class $\eta + \lambda\theta$ with $\lambda > 0$.* □

Because of this result, one defines the group $QFH'(Y,f)$ as the Floer homology of the Lagrangians $\alpha_1 \times \ldots \times \alpha_{g-k}$ and $\beta_1 \times \ldots \times \beta_{g-k}$ in $\mathrm{Sym}^{g-k}(\Sigma)$.

4.2 Proof of Theorem 11

To complete the proof of Theorem 11, we need to show that $HF(V_{\alpha_1}, \ldots, V_{\alpha_{g-k}}, V^t_{\beta_{g-k}}, \ldots V^t_{\beta_1})$ and $HF(V_{\alpha_1} \circ \ldots \circ V_{\alpha_{g-k}}, V^t_{\beta_{g-k}} \circ \ldots \circ V^t_{\beta_1})$ are isomorphic.

For this, we need to recall the calculations of [14] to verify the assumptions (4) in Theorem 3. The area-index relation follows from Lemma 4.17 in [14]. Indeed, in the construction of the Lagrangian correspondences V_{α_i} and V_{β_i} one can choose "strongly admissible" degenerations to ensure that two trajectories with the same index and the same asymptotic limits have the same area. This is a consequence of monotonicity proved in Lemma 4.17 of [14] which implies that there is a well-defined action functional that is S^1-valued. (Alternatively, one can work in "balanced", i.e. "weakly admissible" setting, as in [10] which requires adaptation of the proof of the main theorem to cover this case. We think that this is straightforward but we do not check this here).

Next, in order to ensure *strongly* negative monotonicity condition, that is assumptions (5) in Theorem 3, we need to verify that the index of holomorphic spheres and disks are sufficiently negative as stated in the assumptions of Theorem 3. We show next that this follows from the assumption that $g < 2k$. Note that the symplectic manifolds that we are dealing with are $M = \mathrm{Sym}^n(\Sigma)$ where $n = g(\Sigma) - k$ and $g(\Sigma)$ takes values between $g(\Sigma_{\max}) = g$ and $g(\Sigma_{\min}) = k$. We equip M with a symplectic form in the class $\eta + \lambda\theta$ where $\lambda > 0$ is a fixed parameter that is determined by the monotonicity condition as follows:

The monotonicity constant τ is determined by the equation:

$$[\eta + \lambda\theta] = \tau[c_1(\mathrm{Sym}^n(\Sigma))] = \tau[(n + 1 - g)\eta - \theta]$$

Therefore, $\tau = \frac{1}{n+1-g} < 0$ is the fixed monotonicity constant which is the same for any of the symplectic manifolds we consider since $n - g = -g(\Sigma_{\min})$.

Now, Perutz calculates in Section 4 of [14] that the Hurewicz map $\pi_2(\mathrm{Sym}^n(\Sigma)) \to H_2(\mathrm{Sym}^n(\Sigma))$ has rank 1 and generated by a class h which satisfies $\eta(h) = 1$ and $\theta(h) = 0$. On the other hand, $c_1(\mathrm{Sym}^n(\Sigma)) = (n + 1 - g(\Sigma))\eta - \theta$. Therefore, any simple holomorphic sphere would have $[u] = h$ and its index would be :

$$2(\langle c_1(\mathrm{Sym}^n(\Sigma), h\rangle + n - 3) = 4n - 2g(\Sigma) - 4 = 2g(\Sigma) - 4k - 4$$

The assumption $g < 2k$ now implies that this quantity is strictly less than -4, which suffices for our purpose.

Similarly, for a disk bubble we need to verify the assumptions for our Lagrangian $V_L \subset \mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\Sigma_L)$, where as before $n = g(\Sigma) - k = g(\Sigma_L) + 1 - k$. In light of the fact that Perutz proves in Lemma 3.18 of [13] that any disk in $\pi_2(\mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\Sigma_L), V_L)$ lifts to a sphere it follows that

$$\mu_{V_L}([u]) = 2(\langle c_1(\mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\Sigma_L)), [u]\rangle)$$

Now, the positive area disks u for which the value $\mu_{V_L}([u])$ is maximal, have index given by

$$2(n + 1 - g(\Sigma)) + (2n - 1) - 3 = 2g(\Sigma) - 4k - 2$$

Again, the assumption $g < 2k$ ensures that this value is strictly less than -2, which guarantees that the non-existence of disk bubbles in the $\mathcal{M}^j_J(\underline{x}, \underline{y})$ for $j = 0, 1, 2$.

This completes the verification of the hypothesis of Theorem 3. Therefore, it applies to give the desired isomorphism. $\square$

It is worth pointing out that, the results of [8] now can be put together without the additional hypothesis on figure-eight bubbles to give the isomorphism of $QFH(Y, f; \mathfrak{s})$ with the Heegaard Floer homology groups $HF^+(Y, \mathfrak{s})$.

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